A NOTE ON MECHANICAL RESPONSE IN A PIEZOELECTRIC CERAMIC TRANSDUCER UNDER THE INFLUENCE OF A BODY-FORCE

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ABSTRACT. The present note seeks to investigate the mechanical responses in a piezoelectric transducer made up of a ceramic, when acted upon by a step electrical voltage.

INTRODUCTION

In recent years, there has been a considerable amount of studies on the responses in piezoelectric transducers from the standpoint of mechanics of continuous media, contrary to what has hitherto been followed by means of circuit-theory as in Mason (1948). The recent studies, which are of importance in ultrasonics and acoustics, seem to have been initiated by Redwood (1962), Fillipcyznaski (1956) and followed up by several authors, Sinha (1962, 1963, 1965, 1968), Giri (1966), Roy (1968), Das (1967) and others. In all these discussions, in which responses—both electrical and mechanical—have been studied in a piezoelectric transducer, when voltage or mechanical transients are applied only at one of its ends, the effect of a body-force on its responses has been kept out of consideration. Thus, what is attempted here is to accomodate a suitable body-force, dependent on time as well as dimension, in ascertaining the responses of a piezoelectric transducer. The mechanical response of such a transducer, subjected to a voltage step at one of its ends and rigidly backed at the other end, has been obtained by applying the method of Laplace transforms.

DERIVATION OF THE FUNDAMENTAL EQUATIONS, BOUNDARY CONDITIONS

We consider a transducer vibrating in the thickness mode of vibration. We take this thickness-direction to be the direction of the x-axis so that $x = 0$ and $x = X$ may be taken as the extremities of the transducer. To the end $x = 0$, is applied a voltage transient V given by

$$\begin{aligned} V &= V_0 \text{ when } t > 0 \\ &= 0 \text{ when } t < 0. \end{aligned}$$

The end $x = X$ is rigidly fixed so that the mechanical displacement ξ at $x = X$ is zero. As we seek the mechanical displacement ξ owing to the electrical voltage given by (1), we must necessarily establish a relation between ξ and V. For this purpose, we take the usual constitutive relations for a piezoelectric ceramic as our starting point. These are given by

$$T = c\frac{\partial \xi}{\partial x} - hD \qquad \dots \quad (2)$$

$$E = -h\frac{\partial \xi}{\partial x} + \frac{D}{\epsilon} \qquad \dots \quad (3)$$

where T is the stress, E is the electric intensity and D is the electric displacement in the x-direction. The constants c, h, ϵ represent the elastic compliance, the piezoelectric constant and the dielectric permittivity, respectively.

The equation of motion in the x-direction is given by

$$\rho\,\frac{\partial^2 \xi}{\partial t^2} = \frac{\partial T}{\partial x} + \rho X_1 \qquad \dots \quad (4)$$

where ρ is the density of the material and X_1 is the body-force. Taking X_1 to be given by

$$X_1 = H(t)e^{-kx} \qquad \dots \quad (5)$$

where $H(t)$ is the Heaviside's unit function and k is a constant, we have from (4),

$$\rho\,\frac{\partial^2 \xi}{\partial t^2} = \frac{\partial T}{\partial x} + \rho H(t)e^{-kx} \qquad \dots \quad (6)$$

Because of (2), (6) becomes

$$\rho\,\frac{\partial^2 \xi}{\partial t^2} = c\,\frac{\partial^2 \xi}{\partial x^2} + \rho H(t)e^{-kx} - h\,\frac{\partial D}{\partial x} \qquad \dots \quad (7)$$

Next, in keeping with one simplifying assumption of Redwood (1961) namely, the wave propagation is plane in nature, we get

$$\frac{\partial D_y}{\partial y} = \frac{\partial D_z}{\partial z} = 0$$

which means from

$$\operatorname{div} \vec{D} = 0$$

$$\frac{\partial D}{\partial x} \quad 0. \qquad \dots \quad (8)$$

Hence (7) becomes

$$\rho \frac{\partial^2 \xi}{\partial t^2} = c \frac{\partial^2 \xi}{\partial x^2} + \rho\, H(t) e^{-kx}. \quad \ldots \quad (9)$$

Let $\bar{f}(p)$ be the Laplace transform of $f(t)$ so that

$$\bar{f}(p) = \int_0^\infty e^{-pt} f(t)\, dt, \qquad (p > 0).$$

Taking the Laplace transform of (9), we get

$$\frac{\partial^2 \bar{\xi}}{\partial x^2} - \frac{p^2}{v^2} \bar{\xi} = -\frac{e^{-kx}}{pv^2} \quad \ldots \quad (10)$$

where $$v^2 = \frac{c}{\rho}.$$

Solving for $\bar{\xi}$, we have

$$\bar{\xi} = A e^{-\frac{px}{v}} + B e^{\frac{px}{v}} + \frac{1}{p(p^2 - v^2 k^2)} e^{-kx} \quad \ldots \quad (11)$$

where A and B are constants.

Setting, $F = TYZ$ and $Q = DYZ$

so that F is the force acting on an area of magnitude YZ normal to the x-axis and Q is the electrical charge, we get from (2), using (11),

$$\bar{F} + h\bar{Q} = pZ_c \left\{ -A e^{-\frac{px}{v}} + B e^{\frac{px}{v}} - \frac{vk}{p^2(p^2 - v^2 k^2)} e^{-kx} \right\} \quad \ldots \quad (12)$$

when $$Z_c = \rho v Y Z$$

Integrating the equation (3) between $x = 0$ and $x = X$, we have the voltage V across the transducer in the form given by

$$\bar{V} = -h\{(\bar{\xi})_x - (\bar{\xi})_0\} + \frac{\bar{Q}}{c_0} \quad \ldots \quad (13)$$

where $$c_0 = \frac{\epsilon Y Z}{X}$$

The equations (11), (12), (13) are the fundamental equations. The boundary conditions are nothing but the conditions of continuity of the force and displacement at the extremities $x = 0$ and $x = X$.

SOLUTION OF THE PROBLEM

At $x = 0$, the boundary conditions yield

$$B_1 = A+B+\frac{1}{p(p^2-v^2k^2)} \qquad \dots (14)$$

$$pZ_1B_1 = pZ_c\left\{ -A+B-\frac{vk}{p^2(p^2-v^2k^2)}\right\}-h\bar{Q} \qquad \dots (15)$$

where B_1, Z_1 are the constants corresponding to B and Z_c in the material attached to the transducer at $x = 0$ and where $Z_c = \rho v Y Z$.

At $x = X$, since the plate is rigidly fixed, we get

$$(\bar{\xi})_X = 0$$

which gives from (11),

$$Ae^{-\frac{pX}{v}}+Be^{\frac{pX}{v}}+\frac{1}{p(p^2-v^2k}\, e^{-kX} = 0 \qquad \dots (16)$$

Since the voltage V given by (1) is applied at $x = 0$, we have from (13),

$$h\left\{ A+B+\frac{1}{p(p^2-v^2k^2)}\right\}+\frac{\bar{Q}}{c_0} = \frac{V_0}{p} \qquad \dots (17)$$

The equations (14), (15), (16), (17) can be solved determinantally to evaluate the four unknowns, viz., A, B, B_1, $\bar{Q}$ For this purpose, we may write the above four equations as follows :

$$\begin{aligned}
&A+B-B_1+O.\bar{Q} = -\frac{1}{p(p^2-v^2k^2)}\\
&Z_c.A-Z_c.B+Z_1B_1+h.\frac{\bar{Q}}{p} = -\frac{z\,.vk}{p^2(p^2-v^2k^2)}\\
&e^{-\frac{pX}{v}}.A+e^{\frac{pX}{v}}.B+O.B_1+O.\bar{Q} = -\frac{1}{p(p^2-v^2k^2)}\,e^{-kX}\\
&h.A+h.B+O.B_1+\frac{1}{c_0}.\bar{Q} = \frac{V_0}{p}-\frac{h}{p(p^2-v^2k^2)}
\end{aligned} \qquad (18)$$

Solving for A and B, we get

$$A = \frac{1}{\Delta}\left[-Z_c.\frac{1}{c_0p(p^2-v^2k^2)}\,e^{-kX}-h\left\{e^{\frac{pX}{v}}.\frac{V_0}{p}-\frac{e^{\frac{pX}{v}}}{p(p^2-v^2k^2)}\right\}\right.$$

$$+\frac{h}{p(p^2-v^2k^2)}\,e^{-kX}-\frac{Z_e.vk}{p^2c_0(p^2-v^2k^2)}e^{\frac{pX}{v}}$$

$$+\frac{Z_1}{c_0p(p^2-v^2k^2)}\,e^{-kX}-\frac{Z_1}{p(p^2-v^2k^2)}\cdot\frac{e^{\frac{pX}{v}}}{c_0}\Bigg] \qquad \ldots \quad (19)$$

$$B=\frac{1}{\Delta}\left[\frac{Z_e}{pc_0(p^2-v^2k^2)}e^{-kX}-h\left\{\frac{e^{-\frac{pX}{v}}.V_0}{p}-\frac{e^{-\frac{pX}{v}}}{p(p^2-v^2k^2)}\right\}\right.$$

$$+\frac{he}{p(p^2-v^2k^2)}+\frac{Z_e.vk}{p^2(p^2-v^2k^2)}\cdot\frac{e^{-\frac{pX}{v}}}{c_0}$$

$$\left.+\frac{Z_1}{pc_0(p^2-v^2k^2)}\,e^{-kX}-\frac{Z_1}{p(p^2-v^2k^2)}\,\frac{e^{-\frac{pX}{v}}}{c_0}\right] \qquad \ldots \quad (20)$$

where $\quad \Delta=-e^{-\frac{pX}{v}}\left[\frac{2Z_e}{c_0}\left(1-\frac{h}{c_0}e^{-\frac{pX}{v}}\right)-\frac{(Z_1-Z_v)}{c_0}\left(1-e^{-\frac{2pX}{v}}\right)\right]$

Substituting these values of A and B in (11) and taking its inverse transform, we get the mechanical displacement. In particular, the displacement of the end $x=0$, may be obtained. For, from (11)

$$(\xi)_0=A+B+\frac{1}{p(p^2-v^2k^2)}$$

Therefore, from (19) and (20),

$$(\xi)_0=\frac{1}{\left(\frac{3Z_e}{c_0}-\frac{Z_1}{Z_0}\right)}\left[\left\{\frac{hV_0}{p}-\frac{h}{p(p^2-v^2k^2)}+\frac{Z_1}{p(p^2-v^2k^2)}\cdot\frac{1}{c_0}\right\}\left(1+e^{-\frac{2pX}{v}}\right)\right.$$

$$+\left\{\frac{2h^2}{p(p^2-v^2k^2)}\,e^{-kX}-\frac{2Z_1e^{-kX}}{pc_0(p^2-v^2k^2)}\right\}e^{-\frac{pX}{v}}$$

$$\left.+\left\{\frac{Z_e\cdot vk}{p^2c_0(p^2-v^2k^2)}\left(1-e^{-\frac{2pX}{v}}\right)\right\}\right]$$

$$\times\left[1-r_0e^{-\frac{pX}{v}}-r_1e^{-\frac{2pX}{v}}\right]^{-1}+\frac{1}{p(p^2-v^2k^2)}$$

where

$$r_0 = \frac{h}{3Z_c - Z_1}\ ;\ r_1 = \frac{Z_1 - Z_c}{3Z_c - Z_1}$$

We now expand $\left[1 - r_0 e^{-\frac{pX}{v}} - r_1 e^{-\frac{2pX}{v}}\right]^{-1}$ binomially, as in Redwood (1961) and substitute first few of its terms in (21) and consequently we get on the right of (21) an expression arranged in increasing powers of $e^{-\frac{pX}{v}}$. Considering only the first term of the expansion, we get

$$(\xi)_0 = \frac{1}{\left(\frac{3Z_c}{Z_0} - \frac{Z_1}{Z_0}\right)} \left[\left\{\frac{hV_0}{p} - \frac{h}{p(p^2 - v^2k^2)} + \frac{Z_1}{p(p^2 - v^2k^2)} \cdot \frac{1}{c_0}\right\}\left(1 + r_0 e^{-\frac{pX}{v}}\right)\right.$$

$$\left. + \frac{Z_c . vk}{p^2 c_0 (p^2 - v^2k^2)}\left(1 + r_0 e^{-\frac{pX}{v}}\right)\right] + \frac{1}{p(p^2 - v^2k^2)}$$

where $$r_0 = \frac{h}{3Z_c - Z_1}, \quad r_1 = \frac{Z_1 - Z_c}{3Z_c - Z_1}.$$

Therefore,

$$(\xi)_0 = \frac{1}{\left(\frac{3Z_c}{Z_0} - \frac{Z_1}{Z_0}\right)} \left[\left\{\frac{hV_0}{p} - \frac{h}{p(p^2 - v^2k^2)} + \frac{(pZ_1 + Z_c . vk)}{p^2 c_0 (p^2 - v^2k^2)}\right\}\right.$$

$$\left. \times \left(1 + r_0 e^{-\frac{pX}{v}}\right)\right] + \frac{1}{p(p^2 - v^2k^2)}$$

$$= \left[\frac{1}{\left(\frac{3Z_c}{Z_0} - \frac{Z_1}{Z_0}\right)}\left\{-h + \frac{Z_1}{c_0}\right\} + 1\right] \frac{1}{p(p^2 - v^2k^2)}$$

$$+ \frac{1}{\left(\frac{3Z_c}{Z_0} - \frac{Z_1}{Z_0}\right)} \cdot \frac{hV_0}{p} + \frac{Z_c . vk}{\left(\frac{3Z_c}{Z_0} - \frac{Z_1}{Z_0}\right) c_0} \cdot \frac{1}{p^2(p^2 - v^2k^2)}$$

$$+ \frac{hV_0 r_0}{\left(\frac{3Z_c}{Z_0} - \frac{Z_1}{Z_0}\right)} \cdot \frac{e^{-\frac{pX}{v}}}{p} + \frac{(Z_1 r_0 - h)}{\left(\frac{3Z_c}{Z_0} - \frac{Z_1}{Z_0}\right)} \cdot \frac{1}{(p^2 - v^2k^2)} \cdot \frac{e^{-\frac{pX}{v}}}{p}$$

$$+ \frac{Z_c . vk}{\left(\frac{3Z_c}{Z_0} - \frac{Z_1}{Z_0}\right) c_0} \cdot \frac{1}{p(p^2 - v^2k^2)} \cdot \frac{e^{-\frac{pX}{v}}}{p}.$$

Inverting we get

$$(\xi)_0 = \left[\frac{(Z_1-c_0h_0)}{\left(\frac{3Z_c}{z_0}-\frac{Z_1}{Z_0}\right)c_0}+1\right]\frac{1}{2v^2k^2}\,(e^{vkt}+e^{-vkt}-2)$$

$$+\frac{hV_0}{\left(\frac{3Z_c}{Z_0}-\frac{Z_1}{Z_0}\right)}+\frac{Z_c}{\left(\frac{3Z_c}{Z_0}-\frac{Z_1}{Z_0}\right)c_0}\cdot\frac{1}{v^2k^2}(vkt-\sin vkt),\ 0 < t < \frac{2X}{v}$$

$$= \left[\frac{(Z_1-c_0h_0)}{\left(\frac{3Z_c}{Z_0}-\frac{Z_1}{Z_0}\right)c_0}+1\right]\cdot\frac{1}{2v^2k^2}(e^{vkt}+e^{-vkt}-2)$$

$$+\frac{hV_0}{\left(\frac{3Z_c}{Z_0}-\frac{Z_1}{Z_0}\right)}+\frac{Z_c}{\left(\frac{3Z_c}{Z_0}-\frac{Z_1}{Z_0}\right)c_0}\cdot\frac{1}{v^2k^2}(vkt-\sin vkt)$$

$$+\frac{hV_0r_0}{\left(\frac{3Z_c}{Z_0}-\frac{Z_1}{Z_0}\right)}+\frac{(Z_1r_0-h)}{2\left(\frac{3Z_c}{Z_0}-\frac{Z_1}{Z_0}\right)}\cdot\frac{1}{v^2k^2}\,(e^{vkt}+e^{-vkt}-2)$$

$$+\frac{Z_c\cdot r_0}{2\left(\frac{3Z_c}{Z_0}-\frac{Z_1}{Z_0}\right)c_0vk}(e^{vkt}-e^{-vkt}-2vkt),\ t > \frac{2X}{v}$$

The foregoing results bring out the effect that the body-force, (which is zero when $k\to\infty$), has on the mechanical response of the transducer. The above results are in accordance with those of Redwood (1961) in the absence of the body-force.

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